MATHEMATICAL ANALYSIS OF EQUATION IMITATING DNA BASE OSCILLATIONS

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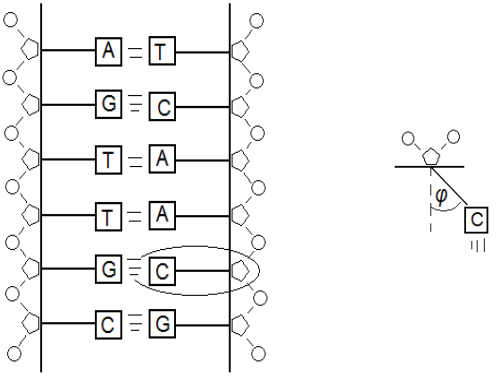
Rotational oscillations of the bases: adenine (*A*)*, thymine* (*T*)*, guanine* (*G*) *and cytosine* (*C*)*, around one of two sugar-phosphate DNA chains are considered in detail. To construct the model equation, an analogy between rotational oscillations of the bases and rotational oscillations of a single mechanical pendulum was used. The equation was analyzed by the method of phase trajectories. As a result, the phase portraits corresponding to different base oscillators and to different models of surrounding conditions have been obtained.*

Introduction. It is widely accepted that DNA is one of the most important biological molecules because of its ability to store and to transfer genetic information. It is also known that DNA structure is not static, but dynamic (Якушевич, 2007). Many investigators even suggest that DNA «breathes». Studies of the DNA «breathing» helps us to understand better the dynamical mechanisms of biological activity of the molecule.

In this work only one aspect of the DNA dynamics is considered, namely, the rotational oscillations of the DNA bases (adenine (A), thymine (T), guanine (G) and cytosine (C)) around the sugar-phosphate chains. We study in detail single base oscillations. To construct model equation, we use the analogy between the rotational oscillations of the DNA bases (Fig. 1a) and the rotational oscillations of a single mechanical pendulum (Awrejcewicz, 1996; Gapa, 2008) (Fig. 1b). At first the analogy has been noticed in the work of Englander and coauthors (Englander et al, 1980), and then it was successfully used by many investigators to model rotational oscillations of the DNA bases.

We obtain and analyze the phase portraits for each base pendulum. At first, we study the phase portraits corresponding to small and large amplitude oscillations with constant forcing F_0 and damping β. Then, we consider the case of periodical external force $F = F_0 \cos \Omega t$.



a b

Fig. 1. Schematic picture of a fragment of double DNA chain including A, T, G and C bases (a) and one base oscillator (b).

Model equation. Taking into account the analogy mentioned above we can suggest that rotational oscillations of a single DNA base is governed by the following equation (Якушевич et al, 2007):

$$I\varphi_{tt}+V\sin\varphi=-\beta\varphi_t+F\,, \tag{1}$$

where $\varphi(t)$ governs the angular displacement of bases, I is the moment of inertia of the bases, V is the parameter determined by the hydrogen interactions between the bases in pairs, β is the coefficient of dissipation, and F is an external generalized force. In this paper we shall consider two simple models of generalized force: constant force ($F = F_0$) and periodical force $F = F_0\cos\Omega t$.

Let us begin with the particular case $F = F_0$. To make calculations easier we can divide both parts of equation (1) by I and apply the following transformation:

$$t=\lambda\tau\,,\ \lambda=10^{-12}\,. \tag{2}$$

Resulting model equation then takes the form:

$$\varphi_{tt}+\omega_0^2\sin\varphi=-b_0\varphi_\tau+k_0\,, \tag{3}$$

where $\omega_0^2=V\lambda^2/I$, $b_0=\beta\lambda/I$, $k_0=F_0\lambda^2/I$.

Equation (3) has three coefficients (ω_0, b_0 and k_0) which take different values for different DNA bases. Estimations of the values carried out with the

help of the data reported in our previous work (Якушевич et al, 2005) are presented in Table 1.

To make mathematical analysis of the model equation, it is convenient (i) to use the system of two differential equations

Table 1. Values of the coefficients of equation (3).

base	ω_0^2	b_0	k_0
A	0.2747	0.0056	0.0041
T	0.2941	0.0087	0.0064
G	0.3797	0.0052	0.0038
C	0.5162	0.0103	0.0076

$$\psi = \varphi_\tau \,, \tag{4}$$

$$\psi_\tau = -\omega_0^2 \sin\varphi - b_0\psi + k_0 \,, \tag{5}$$

which is equivalent to equation (3), and (ii) to construct the phase portraits of the investigated dynamical systems in the plane $\{\varphi,\psi\}$.

Results obtained in the case $\beta = 0$ and $\varphi << 1$. If we suggest that effects of dissipation are absent ($\beta = 0$), the potential energy of a single DNA base oscillator can be written as follows:

$$U(\varphi) = V(1-\cos\varphi) - F_0\varphi \,. \tag{6}$$

After transformation (2) formula (6) takes the form:

$$U(\varphi) = (I/\lambda^2)\cdot u(\varphi) \,, \tag{7}$$

where

$$u(\varphi) = \omega_0^2(1-\cos\varphi) - k_0\varphi \tag{8}$$

plays the role of potential energy of the dynamical system modeled by equation (3).

Potential energy $u(\varphi)$ and phase trajectories in the plane $\{\varphi,\psi\}$ calculated for each of the DNA base oscillators (A, T, G and C) are shown in Fig. 2. When constructing the phase portraits we used the following set of initial conditions: [$\varphi(0) = 0.0$, $\psi(0) = 0.0$], [$\varphi(0) = 0.008$, $\psi(0) = -0.01$], [$\varphi(0) = 0.01$, $\psi(0) = 0.0003$].

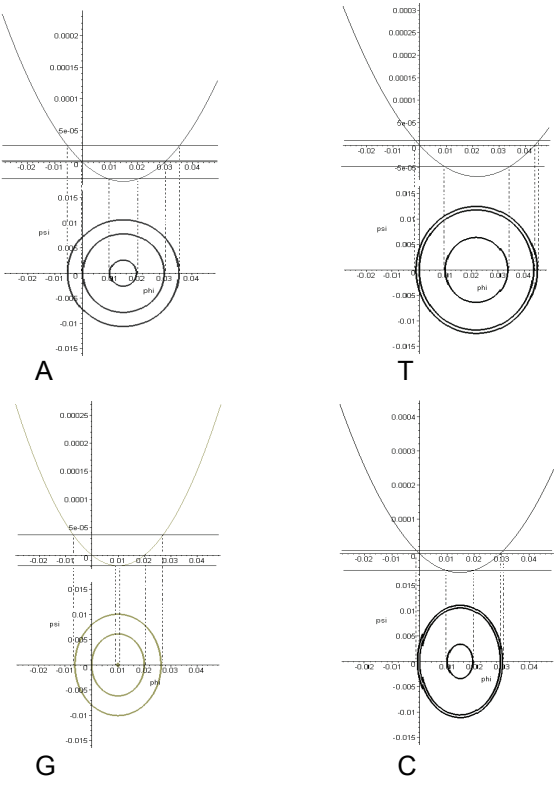


Fig. 2. Potential energy and phase portraits of small amplitude rotational oscillations of adenine (A), thymine (T), guanine (G) and cytosine (C). ($F = F_0, \beta = 0$).

The phase portraits presented in Fig. 2 look like ellipses. The sizes of large and small axes (h_φ and h_ψ) are different for various types of base oscillators. The centers of the ellipses are placed in the points of equilibrium, and they are approximately determined by formulas

$$\varphi_{\min} = k_0 / \omega_0^2 ; \quad (9)$$

$$\psi = 0. \quad (10)$$

Results obtained in the case $\beta \neq 0$ and $\varphi << 1$. If effects of dissipation are taken into account, the phase trajectories calculated for each of four DNA base oscillators (A, T, G and C) are transformed from ellipses (Fig. 2) to spirals (Fig. 3).

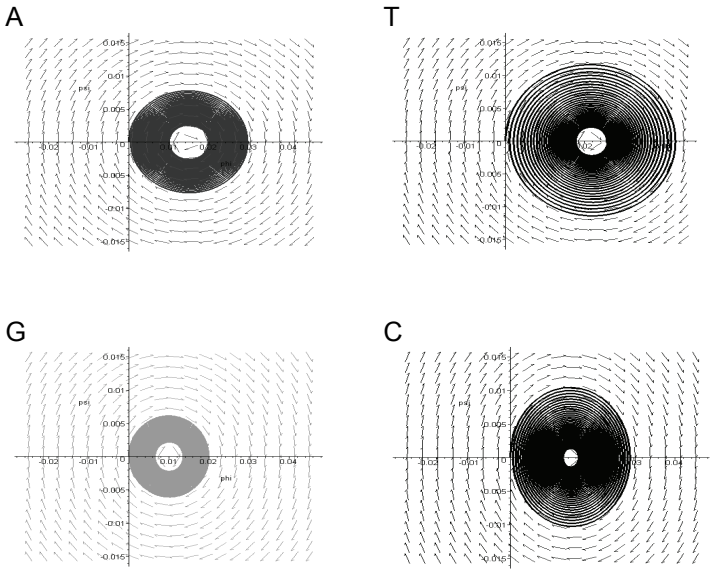


Fig. 3. Phase trajectories of small amplitude rotational oscillations of adenine (A), thymine (T), guanine (G) and cytosine (C). Effects of dissipation and the action of constant generalized force are taken into account.

Each of the portraits shown in Fig. 3 was made for the same initial conditions: $[\varphi(0) = 0.0, \psi(0) = 0.0]$. The phase portraits look like spirals with the centers in the stable equilibrium points approximately determined by formulas (9)–(10). The time of the calculations made to obtain the phase portraits was the same for each of base oscillators. So, the size of holes in Fig. 3 gives us information about the velocity of dissipation.

Results obtained in the case of large amplitudes of oscillations. The figures presented in the previous section were obtained for small amplitudes. In this section we consider large amplitude oscillations. In this case, the

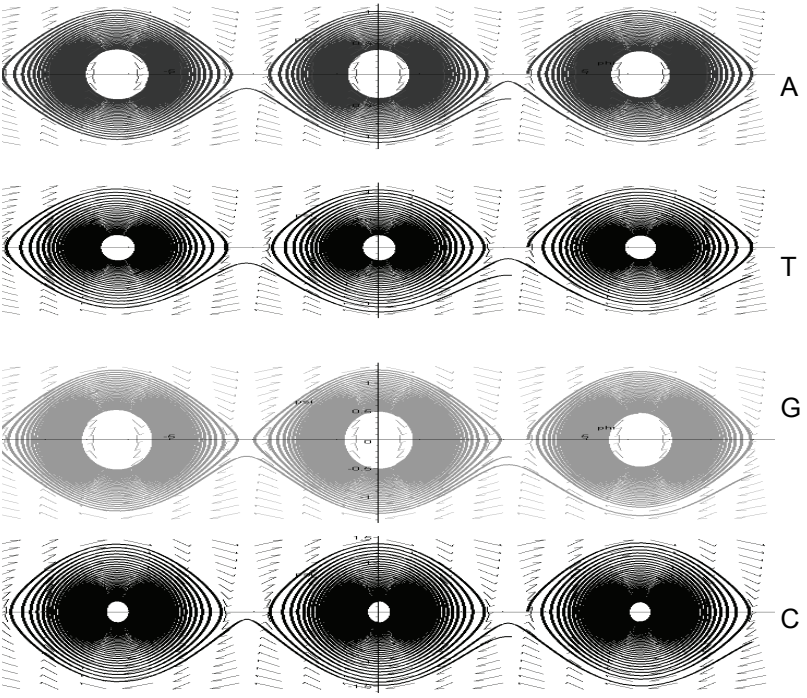


Fig. 4. Phase portraits of large amplitude oscillations of adenine (A), thymine (T), guanine (G) and cytosine (C). Effects of dissipation and the action of constant generalized force are taken into account.

phase trajectories calculated for each of four DNA base oscillators (A, T, G and C) take the form shown in Fig. 4. Each of the portraits was made for the same initial conditions: $[\varphi(0) = 3.2,\ \psi(0) = -0.5]$, $[\varphi(0) = 3.6,\ \psi(0) = 0.01]$, $[\varphi(0) = 9.0,\ \psi(0) = -0.6]$.

Besides general characteristics of the phase portraits discussed above, we can observe an unstable saddle point. Trajectories, which go through that point, mirror the situation when pendulum (base) makes the whole circle around sugar-phosphate chain.

Results obtained in the case of periodical external force. The basic model equation imitating the DNA base oscillations takes in this case the following form:

$$I\varphi_{tt}+V\sin\varphi=-\beta\varphi_t+F_0\cos\Omega t\,. \tag{11}$$

After transformation (2) equation (11) reads

$$\varphi_{tt}+\omega_0^2\sin\varphi=-b_0\varphi_\tau+k_0\cos\gamma\tau\,, \tag{12}$$

where $\gamma=\Omega/\lambda$.

To construct and to analyze the phase trajectories, it is convenient to consider the system of three first order differential equations

$$\psi=\varphi_\tau\,, \tag{13}$$

$$\psi_\tau=-\omega_0^2\sin\varphi-b_0\psi+k_0\cos\theta\,, \tag{14}$$

$$\theta_\tau=\gamma\,, \tag{15}$$

which is equivalent to equation (12). Phase portraits obtained for each of four DNA base oscillators (A, T, G and C) are shown in Fig. 5. To construct trajectories we used the model value γ = 0.5 and the initial conditions [$\varphi(0)$ = 0.0, $\psi(0)$ = 0.0, $\theta(0)$ = 0.0].

In addition to the phase portraits shown in Fig. 5 we obtained the graphs of solution $\varphi(\tau)$ calculated for each of four DNA base oscillators (Fig. 6).

Table 2. Model values of the frequencies ω_0 and γ.

base	ω_0	γ
A	0.5241	0.5
T	0.5423	0.5
G	0.6162	0.5
C	0.7185	0.5

It is easy to notice that the dynamical behavior of the adenine and thymine oscillators substantially differs from that of guanine and cytosine oscillators. It might be explained by the relations between the natural frequency ω_0 and the frequency of periodical external force γ. Indeed, as follows from Table 2, the difference is smaller in the first two cases and bigger in the other two.

After some time, trajectories stabilize and periodical oscillations around equilibrium point with the frequency γ appear. This behavior is illustrated in Fig. 6, where numerical solutions of each pendulum are presented, with the initial conditions [$\varphi(0)$ = 0.0, $\psi(0)$ = 0.0, $\theta(0)$ = 0.0].

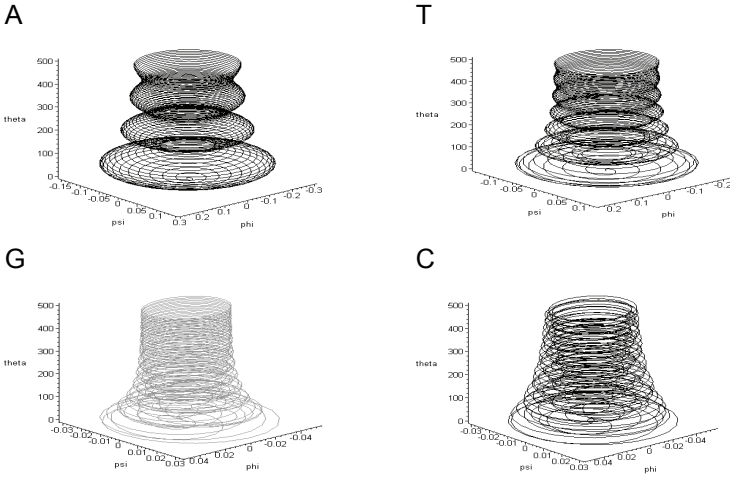


Fig. 5. Phase trajectories of small amplitude oscillations of adenine (A), thymine (T), guanine (G) and cytosine (C) in three dimensional space $\{\varphi,\psi,\theta\}$. Effects of dissipation and the action of periodical generalized force are taken into account.

Conclusions. In this paper we have considered small and large amplitude oscillations of four DNA bases (A, T, G and C). We included into consideration the effects of dissipation and the action of generalized force. Two different models of the force were discussed: constant force F_0 and periodical force $F = F_0 \cos \Omega t$.

We obtained the phase portraits of each of the base oscillators and found the differences in the amplitudes, velocities of dissipation and positions of the equilibrium points.

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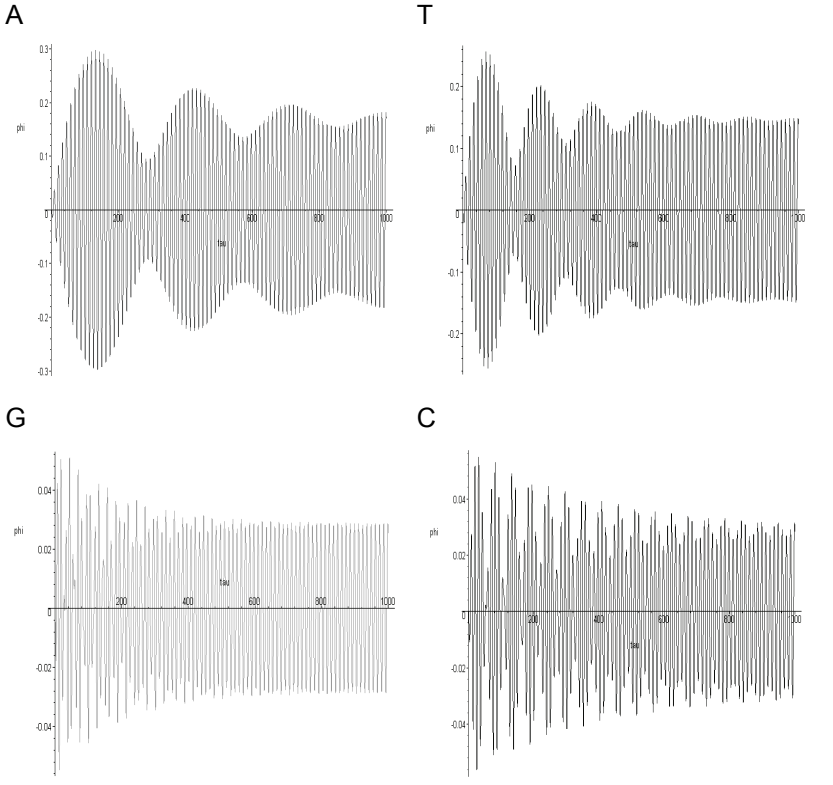

Fig. 6. Small amplitude oscillations $\varphi(\tau)$ of adenine (A), thymine (T), guanine (G) and cytosine (C). Effects of dissipation and the action of periodical generalized force are taken into account.

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МАТЕМАТИЧЕСКИЙ АНАЛИЗ УРАВНЕНИЯ КОЛЕБАНИЙ ОСНОВАНИЙ ДНК

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Исследуются вращательные колебания оснований: аденина (A), тимина (T), гуанина (G) и цитозина (C), вокруг одной из двух сахарофосфатных цепочек ДНК. Для построения модельного уравнения использована аналогия между вращательными колебаниями оснований и вращательными колебаниями математического маятника. Для анализа модельного уравнения был использован метод фазовых траекторий. В результате были получены фазовые портреты осцилляторов, соответствующих различным основаниям ДНК и различным моделям внешнего окружения.